Lecture 10bis. Uniqueness for the Signature of a Path of Bounded Variation

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1 Introduction

Definition 1.1 *If $X_t|_{t\in[0,T]}$ is a bounded 1-variation path in $\mathbb{R}^d$. Then its signature is the sequence of definite iterated integrals*

$$\begin{aligned}\mathbf{X} &= \left(1 + X^1 + \ldots + X^k + \ldots\right)\\ &= \left(1 + \int_{0<u<T} dX + \ldots + \int_{0<u_1<\ldots<u_k<T} dX_{u_1} \otimes \ldots \otimes dX_{u_k} + \ldots\right)\\ &\in T\left((V)\right)\end{aligned}$$

We have already understood that this signature provides a fundamental description of the path X as a control. The goal of this lecture is to determine the equivalence relation ˜ on paths so that

$$X \,\tilde{}\, Y \iff \mathbf{X} = \mathbf{Y}$$

and hence identify the sense in which the signature of a path determines the path.

In [1] Chen proves the following theorems:

Chen Theorem 1: Let $dx_1, \cdots, dx_d$ be the canonical 1-forms on $\mathbb{R}^d$. If $\alpha, \beta \in [a,b] \to R^d$ are sufficiently smooth paths, then the iterated integrals of the vector valued paths $\int_{\alpha(0)}^{\alpha(t)} dx$ and $\int_{\beta(0)}^{\beta(t)} dx$ agree if and only if there exists a translation T of $\mathbb{R}^d$, and a continuous increasing change of parameter $\lambda : [a,b] \to [a,b]$ such that $\alpha = T\beta\lambda$.

Chen Theorem 2: Let G be a Lie group of dimension d, and let $\omega_1 \cdots \omega_d$ be a basis for the left invariant 1-forms on G. If $\alpha, \beta \in [a,b] \to G$ are sufficiently smooth paths, then the iterated integrals of the vector valued paths $\int_{\alpha(0)}^{\alpha(t)} d\omega$ and $\int_{\beta(0)}^{\beta(t)} d\omega$ agree if and only if there exists a translation T of $\mathbb{R}^d$, and a continuous increasing change of parameter $\lambda : [a,b] \to [a,b]$ such that $\alpha = T\beta\lambda$.[1]

We will explain that the signature of a path of bounded 1-variation determines that path up to tree-like extensions and determines it completely as a control and that each equivalence class has a canonical element, the (*tree*) *reduced path* and these reduced paths form a group, a continuous analogue of the free group.

Remark 1.2 *Consider a real tree $T$and a path in $\mathbb{R}^d$ that can be factored into a loop in T and*

[1] We borrow these formulations from the Math Review of the paper.

a projection π of T to $\mathbb{R}^d$.

$$\begin{array}{ccc} [0,K] & \overset{?}{\dashrightarrow} & T \\ & {\scriptstyle x}\searrow & \downarrow{\scriptstyle \pi} \\ & & \mathbb{R}^d \end{array}$$

Such factorisable paths should be negligible for control and, with a slight metric refinement, this picture will be equivalent to the definition of Lipschitz tree-like path we give later and for which we prove the following.

Theorem 1.3 *Let X be a bounded 1-variation path of finite length in $\mathbb{R}^d$. The path X is tree-like if and only if the signature of X is $\mathbf{1} = (1, 0, 0, \dots)$.*

The signatures $\mathbf{X}$, $\mathbf{Y}$ of two paths of finite length are equal if and only if the concatenation of X and 'Y run backwards' is a tree-like path. The 'if' part is essentially trivial given current understanding.

We will concentrate on the harder and more interesting proof that a path with trivial signature is a tree-like path. An immediate consequence of this result is the following.

Corollary 1.4 *The relation $\mathbf{X} \sim \mathbf{Y}$ defined so that two paths of finite length are related if and only if the concatenation of X and 'Y run backwards' is a Lipschitz tree-like path is an equivalence relation.*

Another important consequence will be:

Corollary 1.5 *Given any bounded variation path there exists a unique path of minimal length, called the reduced path, with the same signature.*

There is a sense in which this group of reduced paths is free:

Theorem 1.6 (Cartan development) *Let θ be a linear map of $\mathbb{R}^d$ to the Lie algebra $\mathfrak{g}$ of a Lie group G (the infinitesimal version of a function taking letters to elements of the group) and let $X_t|_{t \leq T}$ be a bounded variation path, then Cartan development provides a canonical projection of $\theta(X)$ to a path Y starting at the origin in G and we can define $\tilde{\theta} : X \to Y_T$. This map $\tilde{\theta}$ is a homomorphism from the space of paths with concatenation to G.*

We might instead consider the space of *geometric p-rough* paths and all the defintions are still meaningfull - all the theorems are open!

Problem 1.7 *Given a path X of finite p-variation for some $p > 1$, is the triviality of the signature of X equivalent to the path being tree-like?*

A basic open question - even in the 1-variation case is:

Problem 1.8 *How does one reconstruct the reduced path from its signature?*

The question is interesting even for paths in the integer lattice. Another very interesting question is to

Problem 1.9 *Identify those elements of the tensor algebra that are signatures of paths and relate properties of the paths (for example their smoothness) to the behaviour of the coefficients.*

2 Tree-Like paths

In this section we work in a more general setting. Suppose that $X_{t\in[0,T]}$ is a path in a Banach or metric space E.

Definition 2.1 *$X_t,\ t\in[0,T]$ is a* tree-like *path if there exists a positive real valued continuous function h defined on $[0,T]$ such that $h(0)=h(T)=0$ and such that*

$$\|X_t - X_s\| \leq h(s) + h(t) - 2\inf_{u\in[s,t]} h(u).$$

The function h will be called a height function for X. We say X is a Lipschitz *tree-like path if h can be chosen to be of bounded variation (i.e. rectifiable).*

Proposition 2.2 *If X is a tree-like path with height function h and, if X is of bounded variation, then there exists a new height function $\tilde{h}$ having bounded variation and hence X is a* Lipschitz *tree like path; moreover, the variation of $\tilde{h}$ is bounded by the variation of X.*

A Lipschitz tree-like path X always has bounded variation less than that of any height function h for X.One can always re-parameterise a continuous path of bounded variation in a finite dimensional Euclidean space so that it is continuous and traversed at speed one. The space of treelike paths is closed.

Lemma 2.3 *Suppose that $\{h_n\}$ are a sequence of height functions on $[0,T]$ for a sequence of tree-like paths $\{X_n\}$. Suppose further that the h_n are parameterised at speeds of at most one and that the X_n take their values in a common compact set within E[2]. Then we may find a subsequence $\left(X_{n(k)}, h_{n(k)}\right)$ converging uniformly to a Lipschitz tree-like path (Y,h). The speed of traversing h is at most one.*

There will alaways be one, and there can be more than one minimiser h for a given X.

3 Approximation of the path

3.1 Representing the path as a line integral against a rank one form

Let γ be a path of finite variation in a finite dimensional Euclidean space E parameterised at unit speed:

$$\omega_\gamma(s,t) = \sup_{\mathcal{D}\subset[s,t]} \sum_{\mathcal{D}} \left\|\gamma_{t_i} - \gamma_{t_{i-1}}\right\| = t - s.$$

We assume that γ has total length T so its parameter set is $[0,T]$. The signature of γ is unaffected by this choice of parameterisation.

Definition 3.1 *Let $\gamma([0,T])$ denote the range of γ in V and let the occupation measure μ on $(V,\ \mathcal{B}(V))$ be denoted*

$$\mu(A) = \left|\{s<T | \gamma(s)\in A\}\right|, \quad A\subset V.$$

Let $n(x)$ be the number of points on $[0,T]$ corresponding under γ to $x\in E$. By the area formulae [5] p125-126, one has the total variation, or length, of the path γ is given by

$$Var(\gamma) = \int n(x)\,\Lambda_1(dx), \tag{3.1}$$

[2] This would be automatic if E were finite dimensional.

where Λ_1 is one dimensional Hausdorff measure. Moreover, for any continuous function f

$$\int f(\gamma(t))\,dt = \int f(x)\,n(x)\,\Lambda_1(dx).$$

Note that $\mu = n(x)\,\Lambda_1$ and that n is integrable.

Lemma 3.2 *The image under γ of a Lesbegue null set is null for μ. That is to say $\mu(\gamma(N)) = \left|\gamma^{-1}\gamma(N)\right| = 0$ if $|N| = 0$.*

Definition 3.3 *We will say that $N \subset [0,T]$ is γ-stable if $\gamma^{-1}\gamma(N) = N$.*

As a result of Lemma 3.2 we see that any null set can always be enlarged to a γ-stable null set.

The Lesbegue differentiation theorem tells us that γ is differentiable at almost every u;the derivative will be absolutely continuous and of modulus one.

Corollary 3.4 *There is a set G of full μ measure in E so that γ is differentiable with $|\gamma'(t)| = 1$ whenever $\gamma(t) \in G$. We set $M = \gamma^{-1}G$. M is γ-stable.*

Now it may well happen that the path visits the same point $m \in M$ more than once. A priori, there is no reason why the directions of the derivative on $\{t \in M | \gamma(t) = m\}$ should not vary. However this can only occur at a countable number of points.

Theorem 3.5 *The set of pairs (s,t) of distinct times in $M \times M$ for which*

$$\begin{aligned} \gamma(s) &= \gamma(t) \\ \gamma'(s) &\neq \gamma'(t) \end{aligned}$$

is countable.

Proof. If (s,t) is such a pair, then $\gamma'(s) \neq \gamma'(t)$. So there is an $\varepsilon > 0$ such that the cones

$$\begin{aligned} |y - \gamma(s)| &< (1+\varepsilon)(y - \gamma(s)).\gamma'(s) \\ |y - \gamma(t)| &< (1+\varepsilon)(y - \gamma(t)).\gamma'(t) \end{aligned}$$

intersect only at $\gamma(s) = \gamma(t)$.

As γ is differentiable with non-zero derivative there is a $\delta_1 > 0$ so that, if $0<|s - \tilde{s}| < \delta_1$, then $\gamma(\tilde{s})$ is in the cone

$$|y - \gamma(s)| < (1+\varepsilon)(y - \gamma(s)).\gamma'(s)$$

and $\gamma(\tilde{s}) \neq \gamma(s)$. Similarly there is a δ_2 so that, if $0<\left|t - \tilde{t}\right| < \delta_2$, then $\gamma\left(\tilde{t}\right)$ is in the cone

$$|y - \gamma(t)| < (1+\varepsilon)(y - \gamma(t)).\gamma'(t),$$

and $\gamma\left(\tilde{t}\right) \neq \gamma(t)$. As a result, $\gamma(\tilde{s}) \neq \gamma\left(\tilde{t}\right)$ or $\gamma(t)$ and $\gamma\left(\tilde{t}\right) \neq \gamma(\tilde{s})$ or $\gamma(s)$.

In particular the pairs (s,t) are isolated in $[0,T] \times [0,T]$ and so are countable in number. ∎

Up to sign and with countably many exceptions, the derivative of γ does not depend the occasion of the visit to a point, only the location. Sometimes we will only be concerned with the unsigned or projective direction of γ and identify $v \in S$ with $-v$.

Definition 3.6 *For clarity we introduce $\tilde{}_{\pm}$ as the equivalence relation that identifies v and $-v$ and let $[\gamma']_{\tilde{}_{\pm}} \in S/\tilde{}_{\pm}$ denote the unsigned direction of γ.*

$\gamma'^{\pm}$ is defined on the full measure subset of $[0,T]$ where γ' is defined and in S.

Corollary 3.7 *There is a function ϕ defined on G with values in the projective sphere $S/\tilde{}_{\pm}$ so that $\phi(\gamma(t)) = [\gamma'(t)]_{\tilde{}_{\pm}}$.*

As a result we may define a very interesting vector valued 1-form μ-almost everywhere on G. If ξ is a vector in S, then $\langle \xi, u\rangle \xi$ is the linear projection of u onto the subspace spanned by ξ. As $\langle \xi, u\rangle \xi = \langle -\xi, u\rangle (-\xi)$ it defines a function from $S/\tilde{}_{\pm}$ to $Hom(V,V)$.

Definition 3.8 *We define the tangential projection 1-form ω. Let ξ be a unit strength vector field on G with $[\xi]_{\tilde{}_{\pm}} = \phi$. Then*

$$\omega(g,u) = \langle \xi(g), u\rangle \xi(g), \quad \forall g \in G, \forall u$$

defines a vector 1-form. The 1-form depends on ϕ, but is otherwise independent of the choice of ξ.

The 1-form ω is the projection of u onto the line determined by $\phi(g)$.

Lemma 3.9 *The tangential projection ω, defined μ a.e. on G, is a linear map from $V \to V$ with rank one. For almost every t one has*

$$\gamma'(t) = \omega(\gamma(t), \gamma'(t))$$

and as a result, using the fundamental theorem of calculus for Lipschitz functions,

$$\begin{aligned}\gamma(t) &= \int_{0<u<t} d\gamma_u + \gamma(0) \\ &\int_{0<u<t} \omega \circ d\gamma_u + \gamma(0),\end{aligned}$$

for every $t \leq T$.

This is a CORE remark. By approximating ω by other rank one 1-forms we will be able to approximate γ by (weakly) piecewise linear paths; we will now prove that *any integral* of our trivial signature path against a one form (even a measurable one) has trivial signature. We have found a way to mollify a path and keep its signature trivial!

Now we have already seen that iterated integrals of iterated integrals are linear functionals of the signature; as a result it is obvious that

Lemma 3.10 *If a path segment has trivial signature, then it has all iterated integrals of iterated integrals are zero.*

In particular polynomials in γ have trivial signature. Now recall that the occupation measure μ of the path γ has finite total mass equal to the length T and that indefinite line integral $y_t := \int_0^t \omega(d\gamma_t)$ is well defined, and continuous as a map from ω in a path in $L^1(V, \mathcal{B}(V), \mu)$ to paths in W with 1-variation. The support of μ is compact, so by the Stone Weierstrass theorem the polynomials are dense in $L^1(V, \mathcal{B}(V), \mu)$. We already know they are paths trivial signature and because the signature is continuous in 1-variation one concludes that

Corollary 3.11 *If γ has trivial signature, then so does $\int \omega(d\gamma_t)$ for any form ω in $L^1(V, \mathcal{B}(V), \mu)$*

Definition 3.12 *A vector valued 1-form ω is (at each point of V) a linear map between vector spaces. We say the 1-form ω is of rank $k \in \mathbb{N}$ on the support of μ if $\dim(\omega(V)) \leq k$ at μ almost every point in V.*

Now any measurable function *is*continuous except on a set of small measure, etc. It is a simple step to approximate our projection form (which is rank one) by piecewise constant rank one forms:

Lemma 3.13 *If ω is a measurable 1-form in $L^1(V, \mathcal{B}(V), \mu)$, then there are finitely many disjoint compact subsets K_i of K and a 1-form $\tilde{\omega}$, that is zero off the K_i and constant on each K_i, such that*

$$\int_K \|\omega - \tilde{\omega}\| \, \mu(dx) \leq \varepsilon$$

and with the property that $\tilde{\omega}$ is rank one if ω is.

4 Piecewise linear paths with no repeated edges.

We call a path γ piecewise linear if it is continuous, and if there is a finite partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_r = T$$

such that γ is linear (or more generally, geodesic) on each segment $[t_i, t_{i+1}]$. Such paths appear naturaly as the integrals of our form $\tilde{\omega}$ we constructed above.

Definition 4.1 *We say the path is nondegenerate if we can choose the partition so that $\left[\gamma_{t_{i-1}}, \gamma_{t_i}\right]$ and $\left[\gamma_{t_i}, \gamma_{t_{i+1}}\right]$ are not colinear for any $0 < i < r$ and if the $\left[\gamma_{t_{i-1}}, \gamma_{t_i}\right]$ are non-zero for every $0 < i \leq r$.*

The positive length condition is automatic if the path is parameterised at unit speed and $0 < T$. If θ_i is the angle $\measuredangle \gamma_{t_{i-1}} \gamma_{t_i} \gamma_{t_{i+1}}$, then γ is non-degenerate if we can find a partition so that for each $0 < i < r$ one has

$$|\theta_i| \neq 0 \bmod \pi.$$

This partition is unique, and we refer to the $\left[\gamma_{t_{i-1}}, \gamma_{t_i}\right]$ as the i-th linear segment in γ.

We now explain that *a nondegenerate piecewise linear path has a non-trivial signature.* To do so we will develop our path into hyperbolic space and use simple hyperbolic geometry.

Fix A (in hyperbolic space), and consider two other points B and C. Let θ_A, θ_B, and θ_C be the angles at A, B, and C respectively. Let a, b, and c be the hyperbolic lengths of the opposite sides. Recall the hyperbolic cosine rule[3]:

$$\sinh(b) \sinh(c) \cos(\theta_A) = \cosh(b) \cosh(c) - \cosh(a)$$

and note the following simple lemmas:

Lemma 4.2 *If the distance c from A to B is at least $\ln\left(\frac{\cos|\theta_A|+1}{1-\cos|\theta_A|}\right)$, then*

$$|\theta_B| \leq |\theta_A|.$$

[3]Our source for this was http://www.maths.gla.ac.uk/~wws/cabripages/hyperbolic/hypertrig.html

Proof. Fix c and the angle θ_A, the angle θ_B is zero if $b = 0$ and monotone increasing as $b \to \infty$. Suppose that $|\theta_B| > |\theta_A|$. We may reduce b so that $|\theta_B| = |\theta_A|$, now the triangle has two equal edges and applying the cosine rule to compute the base length:

$$\begin{aligned}
\sinh(a)\sinh(c)\cos(\theta_A) &= \cosh(a)\cosh(c) - \cosh(a) \\
c &= \ln\left(-\frac{(\cos|\theta_A|)\,e^{2a} + e^{2a} - \cos|\theta_A| + 1}{-e^{2a} + (\cos|\theta_A|)\,e^{2a} - \cos|\theta_A| - 1}\right) \\
&< \lim_{a\to\infty} \ln\left(-\frac{(\cos|\theta_A|)\,e^{2a} + e^{2a} - \cos|\theta_A| + 1}{-e^{2a} + (\cos|\theta_A|)\,e^{2a} - \cos|\theta_A| - 1}\right) \\
&= \ln\left(\frac{\cos|\theta_A| + 1}{1 - \cos|\theta_A|}\right).
\end{aligned}$$

∎

Lemma 4.3 *If* $\max(b,c) \geq \log \frac{2}{1-\cos\theta_A}$, *then* $a > \min(b,c)$.

Proof. Suppose that θ_A is fixed and the triangle has sides $a(\lambda)$, λb, λc. Then

$$\lambda b + \lambda c - a(\lambda)$$

is monotone increasing in λ with a finite limit. Now

$$\begin{aligned}
\sinh(\lambda b)\sinh(\lambda c)\cos(\theta_A) &= \cosh(\lambda b)\cosh(\lambda c) - \cosh(a(\lambda)) \\
\frac{\cosh(\lambda b)\cosh(\lambda c)}{\sinh(\lambda b)\sinh(\lambda c)} - \cos(\theta_A) &= \frac{\cosh(a(\lambda))}{\sinh(\lambda b)\sinh(\lambda c)} \\
\lim_{\lambda\to\infty} \log \frac{\cosh(a(\lambda))}{\sinh(\lambda b)\sinh(\lambda c)} &= \lim_{\lambda\to\infty} (a(\lambda) - \lambda b - \lambda b) + \log 2 \\
\lambda b + \lambda c - a(\lambda) &\leq \lim_{\lambda\to\infty} (\lambda b + \lambda c - a(\lambda)) \\
&= \log \frac{2}{1-\cos\theta_A}.
\end{aligned}$$

Thus

$$a \geq b + c - \log \frac{2}{1-\cos\theta_A}$$

and, providing $\max(b,c) \geq \log \frac{2}{1-\cos\theta_A}$, one has $a \geq \min(b,c)$. ∎

Corollary 4.4 *If the distance* c *from* A *to* B *is at least* $\ln\left(\frac{2}{1-\cos|\theta_A|}\right)$, *then*

$$|\theta_B| \leq |\theta_A|,$$

and $a \geq b$.

Corollary 4.5 *If* X_t *is a continuous piecewise geodesic path of finite length in hyperbolic space with at least one non-trivial geodesic section, and suppose that,*

1. *at each change in direction* t *the angle between the two geodesic segments:* $\angle X_{t-}X_tX_{t+}$ *is at least* $2\theta_A$ *and*

2. that each geodesic segment has length at least $R(\theta_A) = \ln\left(\frac{2}{1-\cos|\theta_A|}\right)$,

then $d(X_0, X_T) \geq R(\theta_A)$ *and the angle between* $\overrightarrow{X_{T-}X_T}$ *and* $\overrightarrow{X_0X_T}$ *is at most* θ_A.

Proof. As the path has finite length and each segment is of length at least $R(\theta_A) > 0$ there can be at most a finite number of distinct piecewise linear segments in the path. We proceed by induction on the number of these geodesic segments in the path.

There is only ever one geodesic through two points in hyperbolic space and so the distance between the ends of a geodesic segment is always the length of the connecting segment. So if there is only one segment we can conclude from 1) that $d(X_0, X_T) \geq R(A)$. The angle between $\overrightarrow{X_{T-}X_T}$ and $\overrightarrow{X_0X_T}$ is zero.

Suppose there are N segments and the penultimate one ends and the last one begins at a time $S < T$. By the induction hypothesis we can assume that the distance $d(X_0, X_S) \geq R(\theta_A)$ and by 2) that $d(X_S, X_T) \geq R(\theta_A)$. Moreover the angle between $\overrightarrow{X_{S-}X_S}$ and $\overrightarrow{X_0X_S}$ is at most θ_A while the angle $X_{S-}X_SX_T$ is at least $2\theta_A$ so that the angle $X_0X_SX_T$ is at least θ_A. We can apply Lemma 4.3 and deduce that

$$d(X_0, X_T) \geq R(\theta_A)$$

and that the angle $X_SX_TX_0$ is at most θ_A. ∎

Corollary 4.6 *Any non-degenerate piecewise linear path* γ *has non-trivial signature.*

Proof. Suppose γ is a non-degenerate piecewise linear path in V and let 2δ be the smallest angle between adjacent edges, and let $D > 0$ denote the length of the shortest edge. Choose $\lambda > R(\delta)/D$.

Now isometrically embed V into the tangent space to a fixed point in hyperbolic space. Then one can consider the development Γ of $\lambda\gamma$ to hyperbolic space. It is a piecewise geodesic path in hyperbolic space with edge lengths greater than $R(\delta)$ and with the angles between any two edges at least 2δ. Thus we can deduce that the distance $|\Gamma(0) - \Gamma(T)|$ is at least $R(\delta) > 0$.

On the other hand, we may recover the same path Γ through solving a linear differential equation in a matrix group. The solution can be expanded into a *convergent* series so that, if the signature of $\lambda\gamma$ were trivial over $[0, T]$, then the development Γ must have $\Gamma(0) = \Gamma(T)$. This is a contradiction. ∎

Corollary 4.7 *Any piecewise linear path* γ *that has trivial signature is tree-like with a height function* h *having the same total variation as* γ.

Proof. We will proceed by induction on the number r of edges in the minimal partition

$$0 = t_0 < t_1 < t_2 < \ldots < t_r = T$$

of γ. We assume that γ is linear on each segment $[t_i, t_{i+1}]$ and that γ is always parameterised at unit speed.

We assume that γ has trivial signature. Our goal is to find a continuous real valued function h with $h \geq 0$, $h(0) = h(T) = 0$, and so that for every s, $t \in [0, T]$ one has

$$\begin{aligned} |h(s) - h(t)| &\leq |t - s| \\ |\gamma_s - \gamma_t| &\leq h(s) + h(t) - 2 \inf_{u \in [s,t]} h(u). \end{aligned}$$

If $r = 0$ the result is obvious; in this case $T = 0$ and the function $h = 0$ does the job.

Now suppose that the minimal partition into linear pieces has $r > 0$ pieces. By Corollary 4.6, it must be a degenerate partition. In other words one of the $\theta_i = \measuredangle\gamma_{t_{i-1}}\gamma_{t_i}\gamma_{t_{i+1}}$ must have

$$|\theta_i| = 0 \bmod \pi.$$

If $\theta_i = \pi$ the point t_i could be dropped from the partition and the path would still be linear. As we have chosen the partition to be minimal this case cannot occur and we conclude that $\theta_i = 0$ and the path retraces its trajectory for an interval of length

$$s = \min\left(|t_i - t_{i-1}|, |t_{i+1} - t_i|\right) > 0.$$

Now $\gamma(t_i - u) = \gamma(t_i + u)$ for $u \in [0, s]$ and either $t_i - s = t_{i-1}$ or $t_i + s = t_{i+1}$. Suppose that the former holds. Consider the path segments obtained by restricting the path to the disjoint intervals

$$\begin{aligned} \gamma_- &= \gamma|_{[0,t_{i-1}]} \\ \gamma_+ &= \gamma|_{[t_i+s,T]} \\ \tau &= \gamma|_{[t_i-s,t_i+s]}, \end{aligned}$$

then $\gamma = \gamma_- * \tau * \gamma_+$ where $*$ denotes concatenation.

The signature $\gamma \to S(\gamma)$ is a function taking path segments with the operation $*$ to sequences of tensors with the operation $\otimes$. It is quite easy to see that it is a homomorphism (c.f. Chen's Identity [3]). As a consequence one sees that the product of the signatures associated to the segments is the signature of the concatenation of the paths and hence is trivial,

$$\begin{aligned} S(\gamma_-) \otimes S(\tau) \otimes S(\gamma_+) &= S(\gamma) \\ &= 1 \oplus 0 \oplus 0 \oplus \ldots \in T(V). \end{aligned}$$

On the other hand the path τ is a linear trajectory followed by its reverse and as reversal produces the inverse signature

$$S(\tau) = 1 \oplus 0 \oplus 0 \oplus \ldots \in T(V).$$

Thus

$$S(\gamma_-) \otimes S(\gamma_+) = 1 \oplus 0 \oplus 0 \oplus \ldots \in T(V)$$

and so the concatenation of γ_- and γ_+ (γ with τ excised) also has a trivial signature. As it is piecewise linear with at least one less edge we may apply the induction hypothesis to conclude that this reduced path is tree-like. Let $\tilde{h}$ be the height function for the reduced path. Then define

$$\begin{aligned} h(u) &= \tilde{h}(u), u \in [0, t_{i-1}] \\ h(u) &= \tilde{h}(u - 2s), u \in [t_i + s, T] \\ h(u) &= s - |t_i - u| + \tilde{h}(t_{i-1}), u \in [t_i - s, t_i + s]. \end{aligned}$$

It is easy to check that h is a height function for γ with the required properties. ∎

Definition 4.8 *A continuous path γ_t is weakly linear (geodesic) on $[0, T]$ if there is a line l (or geodesic l) so that $\gamma_t \in l$ for all $t \in [0, T]$.*

Proposition 4.9 *Any weakly piecewise linear path γ with trivial signature is tree-like with a height function whose total variation is the same as that of γ.*

We now have our main tool. We know how to approximate our path of finite 1-variation by paths that are weakly piecewise linear and at the same time have trivial signatue. They are treelinke and so by a simple compactness argument their limit must be as well!

References

[1] Chen, Kuo-sai T. *Integration of paths—a faithful representation of paths by non-commutative formal power series.* Trans. Amer. Math. Soc. 89 (1958), 395–407

[2] Le Gall, J.F. *Brownian excursions, trees and measure-valued branching processes* Ann. Probab. 19 (1991), 1399–1439

[3] Lyons, T.J. and Qian, Z. *Rough paths and system control*, Oxford University Press, 2003.

[4] Morgan, John W., Shalen, Peter B., *Valuations, trees, and degenerations of hyperbolic structures. I.* Ann. of Math. (2) 120 (1984), 401–476.

[5] Ohtsuka, M. *Dirichlet problems on Riemann surfaces and conformal mappings.* Nagoya Math J., 3 (1951), 91–137.